STABILITY OF NONLINEAR AUTOMATIC CONTROL SYSTEMS

(OB USTOICHIVOSTI NELINEINYKH SISTEM AVTOMATICHESKOGO REGULIROVANIIA)

PMM Vol.23, No.3, 1959, pp. 505-514

O.I. KOMARNITSKAIA
(Leningrad)

(Received 29 October 1958)

There are many methods of stability analysis of nonlinear automatic control systems. As a basis of many investigations in this field the works of Lur'e [3] were used in which he applied a special transformation to equations of motion of nonlinear automatic control systems and indicated a method of constructing Liapunov's function for them. Lur'e's method reduces the problem of stability of equilibrium position in automatic control systems to the investigation of solvability of algebraic systems of quadratic equations. This method gives wide stability regions and is convenient when the number of equations is small. With the increase of degrees of freedom the application of the method becomes more complex because of the difficulty in establishing criteria for the solvability of systems of quadratic equations of higher order.

Malikin [2,4] offered a different method of constructing Liapunov's function for equations of motion of automatic control systems which results in simpler stability conditions.

The method of Malkin is as follows. Let the motion of an automatic control system with one control element be described by differential equations which in terms of canonic variables have the form:

$$\dot{x} = \lambda x + f(\sigma) e \qquad \dot{\sigma} = \beta x - r f(\sigma) \tag{0.1}$$

where x, β, e are column matrices of the n-th order containing elements x_i, β_i, $e_i = 1$, $(i = 1, \ldots, n)$, respectively, and λ is a diagonal matrix with the elements λ_i. The scalars in equation (0.1) have the following meaning: β_i are characteristic constants of the system, λ_i are roots of the characteristic equation with Re $\lambda_i < 0$, $r > 0$ is a constant called the feedback coefficient, σ is a parameter determining position of the control element, and $f(\sigma)$ is a function satisfying the following conditions:

(a) $f(\sigma)$ is continuous and is such that for given initial conditions the equations (0.1) have a unique solution;

(b) $f(0) = 0$, $f(\sigma)\sigma > 0$ for $\sigma \neq 0$. Liapunov's function used is of the form

$$V = \frac{1}{2} Axx + \int_0^\sigma f(\sigma)\, d\sigma \tag{0.2}$$

where A is a symmetric positive definite quadratic form and is derived from the condition

$$\theta = \frac{1}{2}(A\lambda + \lambda' A) \tag{0.3}$$

(λ' is the transposition matrix of λ) and θ is some matrix of the negative-definite quadratic form.

Having calculated the derivative of V on the basis of (0.1) and taking into account the choice of θ, we obtain asymptotic stability conditions for the equilibrium position for the equations (0.1):

$$\Delta = \begin{vmatrix} -\theta & g \\ g' & r \end{vmatrix} > 0 \tag{0.4}$$

where the column matrix g is found from the formula

$$2g = -\beta - Ae \tag{0.5}$$

Malkin does not elaborate on how to select the θ matrix, so that the method will give good practical results; furthermore, he used this method only in the case of different roots of characteristic equation and one zero root of the second order not simple with respect to elementary dividers.

In this paper a form of the θ matrix is given, stability regions are investigated, and Malkin's method is used in the case of multiple zero roots both simple and not simple with respect to elementary dividers.

1. Let us consider an example which will demonstrate how the matrix can be selected and at the same time will permit one to compare the results obtained by means of Malkin's method with results obtained by means of other methods.

Let a nonlinear automatic control system be described by the following equations in canonic form:

$$\dot{x}_1 = \lambda_1 x_1 + f(\sigma), \qquad \dot{x}_2 = \lambda_2 x_2 + f(\sigma), \ \dot{\sigma} = \beta_1 x_1 + \beta_2 x_2 - rf(\sigma)$$

All quantities entering these equations are the same as in equations (0.1) and $\lambda_1 < 0$, $\lambda_2 < 0$ are real. Let us choose the θ matrix of the form

$$\theta = \begin{Vmatrix} -\varepsilon_1 & 0 \\ 0 & -\varepsilon_2 \end{Vmatrix}$$

where ϵ_1 and ϵ_2 are some positive numbers. In accordance with the general theory, the symmetrical matrix A will be found from the condition (0.3). Then

$$a_{11} = -\frac{\varepsilon_1}{\lambda_1}, \qquad a_{12} = 0, \qquad a_{22} = -\frac{\varepsilon_2}{\lambda_2}, \qquad A = \begin{Vmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{Vmatrix}$$

Using (0.5) let us write the matrix $g^{1/2}$

$$g = \begin{Vmatrix} \frac{\varepsilon_1}{2\lambda_1} & -\frac{\beta_1}{2} \\ \frac{\varepsilon_2}{2\lambda_2} & -\frac{\beta_2}{2} \end{Vmatrix}$$

Let us now write the determinant Δ such that

$$\Delta = \begin{vmatrix} \varepsilon_1 & 0 & \frac{\varepsilon_1}{2\lambda_1} - \frac{\beta_1}{2} \\ 0 & \varepsilon_2 & \frac{\varepsilon_2}{2\lambda_2} - \frac{\beta_2}{2} \\ \frac{\varepsilon_1}{2\lambda_1} - \frac{\beta_1}{2} & \frac{\varepsilon_2}{2\lambda_2} - \frac{\beta_2}{2} & r \end{vmatrix} > 0$$

or

$$\frac{1}{\varepsilon_1}\left(\frac{\varepsilon_1}{2\lambda_1} - \frac{\beta_1}{2}\right)^2 + \frac{1}{\varepsilon_2}\left(\frac{\varepsilon_2}{2\lambda_2} - \frac{\beta_2}{2}\right)^2 < r \tag{1.1}$$

This inequality in terms of the rectangular coordinates $1/2\,\beta_1$ and $1/2\,\beta_2$ defines the interior of the ellipse (in accordance with the condition $r > 0$, $\epsilon_1 > 0$, $\epsilon_2 > 0$) whose parameters depend on ϵ_1 and ϵ_2. Let us introduce the notations

$$\frac{\beta_1}{2} = y, \qquad \frac{\beta_2}{2} = z, \qquad \frac{1}{2\lambda_1} = -y_*, \qquad \frac{1}{2\lambda_2} = -z_*$$

Then (1.1) will assume the form:

$$\varepsilon_1^{-1}(y + y_*\varepsilon_1)^2 + \varepsilon_2^{-1}(z + z_*\varepsilon_2)^2 < r \tag{1.2}$$

Utilizing the arbitrariness of ϵ_1 and ϵ_2 let us establish the maximum stability region for given choice of θ.

Let $y + y_*\epsilon_1 = 0$ and $z + z_*\epsilon_2 = 0$ (this can always be obtained by choosing ϵ_1 and ϵ_2); the condition (1.2) will be fulfilled. Then $y = -\epsilon_1 y_*$, $z = -\epsilon_2 z_*$ and since y_*, z_*, ϵ_1, and ϵ_2 are positive (ϵ_1 and ϵ_2, moreover, are arbitrary) y and z can take on any negative values.

Let us now determine what positive values are permissible for y and z.

To this end let us take $y + y_* \epsilon_1 = 0$ so that $\epsilon_2^{-1}(z + z_* \epsilon_2)^2 < r$. From this it follows that

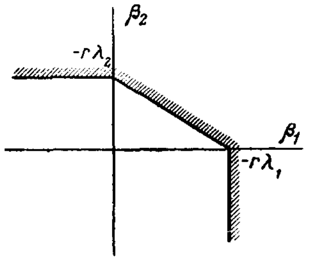

Fig. 1.

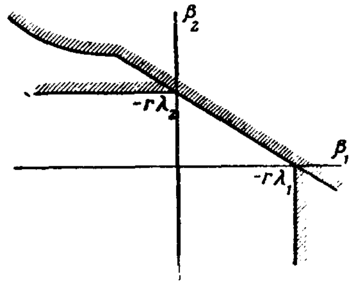

Fig. 2.

$-\sqrt{r\epsilon_1} - z_* \epsilon_2 < z < \sqrt{r\epsilon_2} - z_* \epsilon_2$ and the right-hand side of this inequality reaches its maximum when $\epsilon_2 = r/4z_*^2$. Consequently, (1.2) is satisfied for $y < 0$ and $z < r/4z_*$. In an analogous fashion it can be shown that (1.2) is satisfied for $z < 0$ and $y < r/4y_*$. It remains to determine what will take place when $y > 0$ and $z > 0$. To this end let us introduce a number $k(0 \leqslant k \leqslant r)$ and take $\epsilon_1^{-1}(y + y_* \epsilon_1)^2 < k$; then, using (1.2), we obtain $\epsilon_2^{-1}(z + z_* \epsilon_2)^2 < r - k$. Then

$$y < \frac{k}{4y_*}, \qquad z < \frac{r-k}{4z_*}$$

After eliminating k, we shall have the boundary for positive values of y and z:

$$4y_* y + 4z_* z = r$$

Returning to the original parameters, we shall obtain the following inequalities defining the stability region of the system:

$$\beta_1 < -r\lambda_1, \qquad \begin{cases} \beta_2 < -r\lambda_2 & (\beta_1 < 0) \\ \beta_2 < -r\lambda_2 - (\lambda_2/\lambda_1)\beta_1 & (\beta_1 \geqslant 0) \end{cases}$$

It can be shown that the region of asymptotic stability cannot be increased by changing the θ matrix when a given construction method of the Liapunov function is used. The stability region is depicted in Fig. 1. Let us compare this region with regions obtained for this example by means of other methods.

In Fig. 2 boundaries of regions obtained by this and by Lur'e's methods are shown, and in Fig. 3 this region is compared with the region obtained in [5]. Obviously, the region obtained by this method is narrower than the one obtained by Lur'e's method but it is wider than that obtained by Spasskii. In spite of the fact that other methods give wider stability regions, the method of this paper has the advantage of being more general,

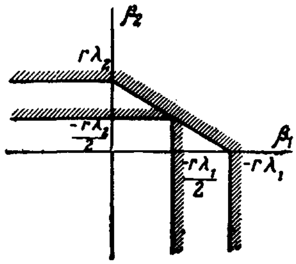

Fig. 3.

i.e. the considerations that are valid for $n = 2$ may be applied for any n. Indeed, let us consider the system (0.1) and let us assume that its characteristic equation has roots which are real negative and all different.

Then (as in the above example) let us take the θ matrix of the form

$$\theta = \left\| \begin{matrix} -\varepsilon_1 & 0 & \ldots & 0 \\ 0 & -\varepsilon_2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & -\varepsilon_n \end{matrix} \right\|$$

where $\epsilon_i > 0$ are arbitrary. Utilizing (0.3) and (0.5) let us find elements of the matrix A and the column matrix g:

$$a_{ii} = -\frac{\varepsilon_i}{\lambda_i}, \qquad a_{ik} = 0\,(i \neq k), \qquad g_i = \frac{\varepsilon_i}{2\lambda_i} - \frac{\beta_i}{2}$$

Let us write the determinant:

$$\Delta = \left| \begin{matrix} \varepsilon_1 & 0 & \ldots & \frac{\varepsilon_1}{2\lambda_1} - \frac{\beta_1}{2} \\ \ldots & \ldots & \ldots & \ldots \\ \frac{\varepsilon_1}{2\lambda_1} - \frac{\beta_1}{2} & \frac{\varepsilon_2}{2\lambda_2} - \frac{\beta_2}{2} & \ldots & r \end{matrix} \right|$$

From the condition $\Delta > 0$ we obtain

$$\frac{1}{\varepsilon_1}\left(\frac{\varepsilon_1}{2\lambda_1} - \frac{\beta_1}{2}\right)^2 + \cdots + \frac{1}{\varepsilon_n}\left(\frac{\varepsilon_n}{2\lambda_n} - \frac{\beta_n}{2}\right)^2 < r$$

After manipulations analogous to the ones of the above example, we shall obtain inequalities defining the stability region for equations (0.1)

$$\beta_i < -\lambda_i r, \qquad \begin{cases} \beta_k < -\lambda_k r & (\beta_i < 0) \\ \beta_k < -\lambda_k (r - S_k) & (\beta_i \geqslant 0) \end{cases} \qquad \left(S_k = \sum_i \frac{\beta_i}{\lambda_i}\right)$$

$$(k = 1, \ldots, n; \quad i = 1, \ldots, k-1, k+1, \ldots, n)$$

This applies to the case of real negative roots.

2. Let us now suppose that the characteristic equation of the system (0.1) has complex conjugate roots with negative real parts as well as real negative roots.

Let λ_i, x_i, $\beta_i (i = 1, \ldots, 2s)$ be complex conjugate pairs, λ_j, x_j, $\beta_j (j = 2s + 1, \ldots n)$ be real numbers; furthermore, $\lambda_j < 0$ and $\mathrm{Re}\,\lambda_i < 0$. In this case let us choose the θ matrix in the form

$$\theta = \left\| \begin{matrix} 0 & -\varepsilon_1 & \ldots & 0 & 0 & 0 & \ldots & 0 \\ -\varepsilon_1 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \ldots & 0 & -\varepsilon_{2s-1} & 0 & \ldots & 0 \\ 0 & 0 & \ldots & -\varepsilon_{2s-1} & 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 & 0 & -2\varepsilon_{2s+1} & \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & -2\varepsilon_n \end{matrix} \right\| \tag{2.1}$$

where $\epsilon_i > 0$ are arbitrary. The same Liapunov's function as before (0.2) will be used and its derivative by virtue of (0.1) will be

$$\dot{V} = \theta xx + (Ae + \beta) xf(\sigma) - rf^2(\sigma) \tag{2.2}$$

The elements of the A matrix determined from (0.3) in this case will be

$$a_{ii} = 0 \quad (i=1, \ldots, 2s), \qquad a_{ii+1} = -\frac{\varepsilon_i}{\lambda_i + \lambda_{i+1}} \quad (i=1,3,5, \ldots, 2s-1)$$

$$a_{ii} = -\frac{\varepsilon_i}{\lambda_i} \quad (i=2s+1, \ldots, n)$$

all other $a_{ik} = 0$.

Let us consider real variables in (2.2):

$$x_k = u_k + u_{k+1} i, \quad \beta_k = \gamma_k + \gamma_{k+1} i$$
$$x_{k+1} = u_k - u_{k+1} i, \quad \beta_{k+1} = \gamma_k - \gamma_{k+1} i \quad (k = 1, 3, \ldots, 2s-1)$$

In terms of the new notations the derivative (2.2) can be written as follows:

$$\dot{V} = -\sum_{k=1,3\ldots}^{2s-1}{}' \varepsilon_k (u_k^2 + u_{k+1}^2) - \sum_{k=2s+1}^{n} \varepsilon_k x_k^2 + \sum_{k=2s+1}^{n} \left(\beta_k - \frac{\varepsilon_k}{\lambda_k}\right) x_k f(\sigma) +$$
$$+ 2 \sum_{k=1}^{s} \left[\left(\gamma_{2k-1} - \frac{\varepsilon_{2k-1}}{\lambda_{2k-1} + \lambda_{2k}}\right) u_{2k-1} + \gamma_{2k} u_{2k}\right] f(\sigma) - rf^2(\sigma)$$

From this, in accordance with (0.4), we shall obtain the following condition for asymptotic stability:

$$\Delta = \begin{vmatrix} \varepsilon_1 & 0 & 0 & \ldots & 0 & \gamma_1^* \\ 0 & \varepsilon_1 & 0 & \ldots & 0 & \gamma_2^* \\ 0 & 0 & \varepsilon_3 & \ldots & 0 & \gamma_3^* \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \ldots & \varepsilon_n & \beta_n^* \\ \gamma_1^* & \gamma_2^* & \gamma_3^* & \ldots & \beta_n^* & r \end{vmatrix} > 0$$

Here

$$\gamma_1^* = \frac{\varepsilon_1}{\lambda_1 + \lambda_2} - \gamma_1, \qquad \gamma_2^* = -\gamma_2,$$

$$\gamma_3^* = \frac{\varepsilon_3}{\lambda_3 + \lambda_4} - \gamma_3, \ldots, \qquad \beta_n^* = \frac{\varepsilon_n}{2\lambda_n} - \frac{\beta_n}{2}$$

or

$$\frac{1}{\varepsilon_1}\left(\frac{\varepsilon_1}{\lambda_1 + \lambda_2} - \gamma_1\right)^2 + \frac{\gamma_2^2}{\varepsilon_1} + \cdots + \frac{1}{\varepsilon_{2s-1}}\left(\frac{\varepsilon_{2s-1}}{\lambda_{2s-1} + \lambda_{2s}} - \gamma_{2s-1}\right)^2 + \\ + \frac{\gamma_{2s}^2}{\varepsilon_{2s-1}} + \cdots + \frac{1}{\varepsilon_n}\left(\frac{\varepsilon_n}{2\lambda_n} - \frac{\beta_n}{2}\right)^2 < r$$

From this, utilizing the arbitrariness of ϵ_1 and ϵ_2, the conditions for asymptotic stability of the system can be obtained in a straightforward fashion:

$$\beta_i < -\frac{m_i \lambda_i}{2}, \qquad \begin{cases} \gamma_{j+1}^2 < l_j(\lambda_j + \lambda_{j+1})\gamma_j + \frac{1}{4}(\lambda_j + \lambda_{j+1})^2 l_j^2 & \beta_i < 0 \\ \gamma_{2k+2}^2 < \left(r - \sum_j l_j\right)(\lambda_{2k+1} + \lambda_{2k+2})\gamma_{2k+1} + \\ \qquad + \frac{1}{4}(\lambda_{2k+1} + \lambda_{2k+2})^2\left(r - \sum_j l_j\right)^2 & \\ \gamma_{j+1}^2 < l_j(\lambda_j + \lambda_{j+1})\gamma_j + \frac{1}{4}(\lambda_j + \lambda_{j+1})^2 l_j^2 & \beta_i \geqslant 0 \\ \gamma_{2k+2}^2 < \left(r - \sum_j l_j - \sum_i m_i\right)(\lambda_{2k+1} + \lambda_{2k+2})\gamma_{2k+1} + \\ \qquad + \frac{1}{4}(\lambda_{2k+1} + \lambda_{2k+2})^2\left(r - \sum_j l_j - \sum_i m_i\right)^2 & \end{cases}$$

$$(i = 2s+1, \ldots, n; \quad j = 1, 3, \ldots, 2k-1, 2k+3, \ldots, 2s-1; \quad k = 1, \ldots, s-1)$$

Here

$$l_j = -\frac{2}{\lambda_j + \lambda_{j+1}}\left(\gamma_j + \sqrt{\gamma_j^2 + \gamma_{j+1}^2}\right), \qquad m_i = -\frac{2\beta_i}{\lambda_i}$$

are positive numbers such that

$$\sum_i m_i + \sum_j l_j \leqslant r$$

Example. Let us consider an automatic control system described by the differential equations which in canonic variables have the following form:

$$\dot{x}_1 = \lambda_1 x_1 + f(\sigma) \qquad \left(\begin{array}{l}\lambda_1 = \mu_1 + \mu_2 i \\ \lambda_2 = \mu_1 - \mu_2 i\end{array}\right)$$
$$\dot{x}_2 = \lambda_2 x_2 + f(\sigma)$$
$$\dot{\sigma} = \beta_1 x_1 + \beta_2 x_2 - rf(\sigma) \quad (\beta_1 = \gamma_1 + \gamma_2 i, \ \beta_2 = \gamma_1 - \gamma_2 i).$$

Here $\mu_1 < 0$, μ_2 and $\gamma_2 \neq 0$, and for all other quantities the same assumptions as in the general case apply.

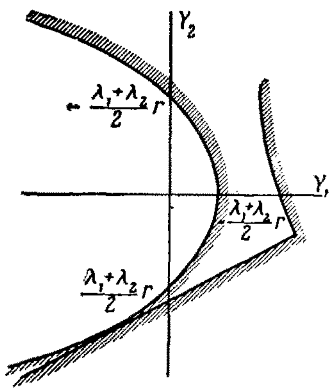

Fig. 4.

In accordance with the derived formulas the stability region for this example will be defined by the following inequality:

$$\gamma_2^2 < r(\lambda_1 + \lambda_2)\gamma_1 + \frac{1}{4}(\lambda_1 + \lambda_2)^2 r^2$$

The stability region is shown in Fig. 4; here, as in the case of real roots of the characteristic equation, the region obtained by the method of Lur'e is wider then the one obtained by this method, but the region of this method can be obtained for any number of degrees of freedom.

3. Let us now consider equations of an automatic control system with the characteristic equation having n different roots $\lambda_1, \ldots, \lambda_n$ such that $\operatorname{Re}\lambda_i < 0$ and the root $\lambda = 0$ of the second order with k groups of corresponding solutions, i.e. the root $\lambda = 0$ repeats k times and every time it has a multiplicity of the second order with respect to elementary dividers.

In terms of canonic variables (and taking into consideration the nature of the roots described above) the equations shall be written as follows:

$$\text{I}\quad\begin{cases}\dot{x}_1=\lambda_1 x_1+\sum\limits_{s=1}^{m}\beta_{s1}f_s(\sigma_s)\\ \cdots\cdots\cdots\\ \dot{x}_n=\lambda_n x_n+\sum\limits_{s=1}^{m}\beta_{sn}f_s(\sigma_s)\end{cases}\qquad \text{II}\quad\begin{cases}\dot{y}_1=\sum\limits_{s=1}^{m}\gamma_{s1}f_s(\sigma_s)\\ \dot{y}_2=y_1+\sum\limits_{s=1}^{m}\gamma_{s2}f_s(\sigma_s)\\ \cdots\cdots\cdots\\ \dot{y}_{2k}=y_{2k-1}+\sum\limits_{s=1}^{m}\gamma_{s2k}f_s(\sigma_s)\end{cases}\qquad(3.1)$$

$$\sigma_s=p_{s1}x_1+\cdots+p_{sn}x_n+q_{s1}y_1+\cdots+q_{s2k}y_{2k}$$

Such roots of the characteristic equation may occur, for instance, in a case when an automatic control system has as its independent elements some bodies that are caused to rotate by the regulator about a fixed axis.

The uniqueness of the equilibrium position is established by the following theorems.

Theorem 1. In order for the equilibrium position to be unique it is necessary that the number of the regulating elements shall not be less than the number of pairs of zero roots ($m \geqslant k$)(number of independent elements).

Theorem 2. If the number of the regulating elements is not less than the number of pairs of zero roots ($m > k$) then in order to ensure that the equilibrium position is unique it is sufficient that the order of the matrices

$$\begin{Vmatrix}\gamma_{11} & \gamma_{21} & \cdots & \gamma_{m1}\\ \gamma_{13} & \gamma_{23} & \cdots & \gamma_{m3}\\ \cdots & \cdots & \cdots & \cdots\\ \gamma_{12k-1} & \gamma_{22k-1} & \cdots & \gamma_{m2k-1}\end{Vmatrix}\qquad\begin{Vmatrix}q_{12} & q_{14} & \cdots & q_{12k}\\ q_{22} & q_{24} & \cdots & q_{22k}\\ \cdots & \cdots & \cdots & \cdots\\ q_{m2} & q_{m4} & \cdots & q_{m2k}\end{Vmatrix}$$

is equal to k.

Theorem 3. If the number of the regulating elements is equal to the number of pairs of zero roots, then in order for the equilibrium position to be unique it is necessary and sufficient that the determinants

$$\gamma=\begin{vmatrix}\gamma_{11} & \gamma_{21} & \cdots & \gamma_{k1}\\ \gamma_{13} & \gamma_{2k} & \cdots & \gamma_{k3}\\ \cdots & \cdots & \cdots & \cdots\\ \gamma_{12k-1} & \gamma_{22k-1} & \cdots & \gamma_{k2k-1}\end{vmatrix}\qquad q=\begin{vmatrix}q_{12} & q_{14} & \cdots & q_{12k}\\ q_{22} & q_{24} & \cdots & q_{22k}\\ \cdots & \cdots & \cdots & \cdots\\ q_{k2} & q_{k4} & \cdots & q_{k2k}\end{vmatrix}$$

are different from zero.

The proofs of all three theorems are similar to each other. Therefore, only the proof of the third theorem will be given here.

At the equilibrium $x_i = c_i$ and $y_j = d_j$ $(i = 1, \ldots, n,\ j = 1, \ldots, 2k)$ where c_i and d_j are some constants. Let us substitute these values into (3.1).

The second group (II) of these equations (3.1) will be satisfied for $\sigma_s = 0$ $(s = 1, \ldots, m)$ and for $d_1 = d_3 = \ldots = d_{2k-1} = 0$.

The first group of these equations (I) is satisfied for $\sigma_s = 0$ and $c_1 = \ldots = c_n = 0$. Taking this into account we shall obtain

$$q_{s2}d_2 + q_{s4}d_4 + \cdots + q_{s2k}d_{2k} = 0 \qquad (s=1, \ldots, m)$$

If one assumes that the determinant $q = 0$, then the latter system permits a non-zero solution and, consequently, the equilibrium position is not unique. Analogously, it can be proved that $\gamma \neq 0$. Thus the restrictions imposed by the theorem are necessary. To prove the sufficiency of the restrictions let us note that for $\gamma \neq 0$ odd equations of the second group (II) of (3.1) with $x_i = c_i$ $y_j = d_j$ will be satisfied only when $f_s(\sigma_s) = 0$. Let us substitute $f_s(\sigma_s) = 0$ into the rest of equations (3.1) and obtain that $c_1 = \ldots = c_n = 0$, $d_1 = d_3 = \ldots = d_{2k-1} = 0$. Furthermore, from the condition $f_s(\sigma_s) = 0$ it follows that all $\sigma_s = 0$, i.e. it follows that

$$q_{s2}d_2 + q_{s4}d_4 + \cdots + q_{s2k}d_{2k} = 0$$

Then from the condition $q \neq 0$ it follows that $d_2 = d_4 = \ldots = d_{2k} = 0$ which proves the sufficiency.

Let us suppose that the number of regulating elements in equations (3.1) is equal to the number of pairs of zero roots and that the Theorem 3 is satisfied. To investigate stability of the system let us write (3.1) in matrix form:

$$\dot{x} = \lambda x + BF, \qquad \dot{y} = Jy + \Gamma F, \qquad \sigma = px + Qy \qquad (3.2)$$

Obviously, the first equation of (3.2) represents the group of equations (3.1) and the second equation represents the second group of (3.1).

The structure of the matrices entering into (3.2) is obvious. They are all of the order of the highest numbers n and $2k$ and all missing elements are filled in with zeros.

Let us construct Liapunov's function for the equations (3.2) in the form

$$V = \frac{1}{2} Axx + \frac{1}{2}\Delta yy + \sum_{s=1}^{m}\int_0^{\sigma_s} f_s(\sigma_s)\,d\sigma_s$$

where the symmetric matrix A is found from the condition (0.3) for θ of the form (2.1) and the matrix Λ is of the form

$$\Delta = \begin{Vmatrix} \delta_1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & \delta_3 & \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \ldots & 0 \end{Vmatrix} \qquad (\delta_i > 0)$$

It is easily seen that V is a positive definite form of the variables $x_1, \ldots, x_n; y_1, \ldots, y_{2k}$. By virtue of the system (3.2) we have

$$\dot V = \theta x\cdot x + \frac{1}{2}[J'\Delta + \Delta J]y\cdot y + \frac{1}{2}[\Gamma'\Delta + QJ]\,y\cdot F + $$
$$+ \frac{1}{2}[B'A + P\lambda]\,x\cdot F + [PB + Q\Gamma]\,F\cdot F$$

It is obvious that $J'\Lambda + \Lambda J \equiv 0$. Let us form the discriminant of the quadratic form $-\dot V$

$$\begin{vmatrix} -\theta & -\frac{1}{4}[P\lambda + B'A] \\ -\frac{1}{4}[P\lambda + B'A] & -\frac{1}{2}[(PB + Q\Gamma) + (PB + Q\Gamma)'] \end{vmatrix} \qquad (3.3)$$

Let us require that the diagonal minors of this determinant from $n + 1$ order and up shall be greater than zero and, furthermore,

$$\Gamma'\Delta + QJ = 0 \qquad (3.4)$$

The last condition contains $2k$ exact equalities but since matrix Λ contains k arbitrary $\delta_i > 0$ there will be only k equalities.

If the above conditions are satisfied, the quadratic form

$$\dot V = \theta x\cdot x + \frac{1}{2}[B'A + P\lambda]\,x\ F + [PB + Q\Gamma]\,FF$$

will be negative definite with respect to the variables x and F and it is always negative with respect to the variables x, y, and F. Nevertheless, it can be shown that if (3.3) and (3.4) are fulfilled, then the equilibrium position of the system (3.2) will be asymptotically stable. To this end it is sufficient to verify that the integral curves of equations (3.2) intersect all hypersurfaces V = const from outside, i.e. that $\dot V$ is always negative or if it becomes zero at some point (other than the origin) then at the next point it will be negative again. Indeed, the always negative form $\dot V$ becomes zero on the line of intersection of hypersurfaces $\sigma_s = 0$ when $x = 0$.

Let us take a point on this line with $x = 0$ and all $y_1 = y_3 = \ldots = y_{2k-1} = 0$. Then, having substituted these values into the system $\sigma_s = 0$ $(s = 1, \ldots, m)$ with $q \neq 0$, we obtain $y_2 = y_4 = \ldots = y_{2k} = 0$, i.e. in this case the point under consideration is the origin. Now let us take some point on the line of intersection such that $x = 0$ and there is at least one $y_i \neq 0$ $(i = 1, 3, \ldots, 2k-1)$. At this point we have

$$\dot{\sigma}_s = q_{s1}y_1 + q_{s3}y_3 + \cdots + q_{s2k-1}y_{2k-1} \qquad (s = 1, \ldots, m)$$

Since $q \neq 0$ and at least one $y_i \neq 0$ then at least one of $\sigma_s(y_i)$ will be other than zero. If $\sigma_s(y_i)$ is not equal to zero then as t increases σ_l will become other than zero, i.e. the integral curve will leave the line of intersection and V will become negative; thus the integral curve has touched a point on the hypersurface $V = \text{const}$, but after this it intersects the surface from outside. From this it is seen that the stability of the equilibrium position is asymptotic.

Example. Let us consider differential equations of motion of a nonlinear automatic system

$$\begin{aligned}
\dot{y}_1 &= \gamma_{11}f_1(\sigma_1) + \gamma_{21}f_2(\sigma_2)\\
\dot{y}_2 &= y_1 + \gamma_{12}f_1(\sigma_1) + \gamma_{22}f_2(\sigma_2)\\
\dot{y}_3 &= \gamma_{13}f_1(\sigma_1) + \gamma_{23}f_2(\sigma_2)\\
\dot{y}_4 &= y_3 + \gamma_{14}f_1(\sigma_1) + \gamma_{24}f_2(\sigma_2)\\
\sigma_1 &= q_{11}y_1 + q_{12}y_2 + q_{13}y_3 + q_{14}y_4\\
\sigma_2 &= q_{21}y_1 + q_{22}y_2 + q_{23}y_3 + q_{24}y_4
\end{aligned}$$

Let us assume that the equilibrium position of the system under consideration is unique and that the restrictions of the Theorem 3 are fulfilled, i.e.

$$\begin{vmatrix} q_{12} & q_{14}\\ q_{22} & q_{24}\end{vmatrix} \neq 0, \qquad \begin{vmatrix} \gamma_{11} & \gamma_{21}\\ \gamma_{13} & \gamma_{23}\end{vmatrix} \neq 0$$

In accordance with the obtained results the conditions for asymptotic stability of the system under consideration may be written as follows:

$$\sum_{k=1}^{4} q_{1k}\gamma_{1k} < 0, \quad \begin{vmatrix} \sum\limits_{k=1}^{4} q_{1k}\gamma_{1k} & \frac{1}{2}\sum\limits_{k=1}^{4}(q_{1k}\gamma_{2k} + q_{2k}\gamma_{1k})\\ \frac{1}{2}\sum\limits_{k=1}^{4}(q_{1k}\gamma_{2k} + q_{2k}\gamma_{1k}) & \sum\limits_{k=1}^{4} q_{2k}\gamma_{2k}\end{vmatrix} > 0$$

$$\gamma_{11}q_{12} < 0, \qquad \gamma_{13}q_{14} < 0, \qquad q_{24}\gamma_{13} - q_{14}\gamma_{23} = 0$$
$$\gamma_{21}q_{22} < 0, \qquad \gamma_{23}q_{24} < 0, \qquad q_{12}\gamma_{21} - \gamma_{11}q_{22} = 0$$

The first two inequalities are obtained from (3.3) and all the others from (3.4) after eliminating $\delta_i > 0$.

4. Let us now consider the equation of a direct automatic control system with m regulating elements and with the characteristic equation having n different roots $\lambda_i (i = 1, \ldots, n)$ such that $\operatorname{Re} \lambda_i < 0$ and the root $\lambda = 0$ of the multiplicity m is simple with respect to elementary dividers

$$\dot{x} = \lambda x + BF, \qquad \dot{y} = \Gamma F, \qquad \sigma = px + Qy \tag{4.1}$$

All matrices entering into (4.1) are of the order of maximum numbers n and m, and their structure is obvious.

Let us suppose that in equations (4.1) the number of regulating elements is equal to the number of zero roots, and that the conditions for uniqueness of the equilibrium position [5] for (4.1) are fulfilled, i.e. $|\Gamma| \neq 0$, $|Q| \neq 0$. Having applied the above-described method of Liapunov's function construction to the system (4.1), it is possible to simplify stability criteria obtained for analogous systems [5].

Let us take Liapunov's function for (4.1) of the form

$$V = \frac{1}{2} Axx + \sum_{s=1}^{m} \int_0^{\sigma_s} f_s(\sigma_s)\, d\sigma_s$$

where A is determined from the condition (0.3); the quadratic form V is positive definite since $|Q| \neq 0$.

Let us differentiate V, taking into account (4.1)

$$\dot{V} = \theta xx + [PB + Q\Gamma] FF + [B'A + P\lambda] xF$$

Let us introduce the notation $PB + Q\Gamma = -R$ and form a discriminant of the last quadratic form $-\dot{V}$:

$$\begin{vmatrix} -\theta & -\frac{1}{2}[B'A + P\lambda] \\ -\frac{1}{2}[B'A + P\lambda] & \frac{1}{2}[R + R'] \end{vmatrix} \tag{4.2}$$

To realize the asymptotic stability of the equilibrium position it is sufficient to require that the diagonal minors of the determinant (4.2) beginning with the $n + 1$ order shall be greater than zero.

Thus, the asymptotic stability conditions are expressed through inequalities, which govern the parameters of the system. In [5], among the stability conditions for this case, there are used identities, which is undesirable in practical applications.

Example. Let us consider equations of motion of an automatic control system

$$\dot{y}_1 = \gamma_{11} f_1(\sigma_1) + \gamma_{21} f_2(\sigma_2), \qquad \sigma_1 = q_{11} y_1 + q_{12} y_2$$
$$\dot{y}_2 = \gamma_{12} f_1(\sigma_1) + \gamma_{22} f_2(\sigma_2), \qquad \sigma_2 = q_{21} y_1 + q_{22} y_2$$

The characteristic equation of this system has a zero root of the second order.

Asymptotic stability conditions in this case will be written as follows:

$$q_{11}\gamma_{11} + q_{12}\gamma_{12} < 0$$

$$\begin{vmatrix} q_{11}\gamma_{11} + q_{12}\gamma_{12} & q_{11}\gamma_{21} + q_{12}\gamma_{22} \\ q_{21}\gamma_{11} + q_{22}\gamma_{12} & q_{21}\gamma_{21} + q_{22}\gamma_{22} \end{vmatrix} > 0$$

These conditions are simpler and more inclusive than those obtained in [5].

BIBLIOGRAPHY

1. Voznesenskii, N.I , K voprosu o vybore skhemy regulirovaniia teplofikatsionnykh turbin (On the question of selection of control schemes for thermal turbines). *Trud. Leningradskogo politekhnicheskogo in-ta* No. 2, 1948.

2. Letov, A.M., *Ustoichivost' nelineinykh reguliruemykh sistem (Stability of Nonlinear Controlled Systems)*. Gostekhizdat, 1955.

3. Lur'e, A.I., *Nekotorye nelineinye zadachi teorii avtomaticheskogo regulirovaniia (Some Nonlinear Problems of Automatic Control Theory)*. Gostekhizdat, 1951.

4. Malkin, I.G., K teorii ustoichivosti reguliruemykh system (On stability theory of controlled systems). *PMM* Vol. 25, No. 1, 1954.

5. Spasskii, P.A., Ob odnom klasse reguliruemykh sistem (On a class of controlled systems). *PMM* Vol. 18, No. 3, 1954.

Translated by P.N.B.